Quantum cohomology of Grassmannians

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Abstract

We recall definition of (small) quantum cohomology of Grassmannians following [Be], give technical details and then give elementary proofs of the main theorems about the quantum cohomology of Grassmannians following Buch's paper ([Bu]). Namely, we prove quantum Giambelli and quantum Pieri formulas and the presentation of quantum cohomology ring.

1 Recollections on cohomologies of Grassmanninans

1.1 Main definitions

Let us fix some notations. Pick $l, n \in \mathbb{Z}_{\geqslant 0}$, $l \leqslant n$ and let V be a vector space of dimension n over complex numbers. We denote by $\mathrm{Gr}(l, V)$ the Grassmanian parametrizing l-dimensional subspaces $W \subset V$. We denote by $\iota\colon \mathrm{Gr}(l, V) \hookrightarrow \mathbb{P}(\Lambda^l(V))$ the Plüker embedding which sends $W \subset V$ to $\Lambda^l(W) \in \mathbb{P}(\Lambda^l(V))$. One can show that $\mathrm{Gr}(l, V)$ is a complex projective algebraic variety of dimension lk, here $k := n - l$.

Note that we have the natural left action $\mathrm{GL}(V) \curvearrowright \mathrm{Gr}(l, V)$: element $g \in \mathrm{GL}(V)$ sends $W \in \mathrm{Gr}(l, V)$ to $g(W) \in \mathrm{Gr}(l, V)$. It is clear that this action is transitive. Let us fix any point $U \in \mathrm{Gr}(l, V)$ and denote by $P \subset \mathrm{GL}(V)$ the stabilizer of U. Then we have the natural identification $G/P \xrightarrow{\sim} \mathrm{Gr}(l, V)$, $g \mapsto g(U)$ which we will use later to define the tautological bundle on $\mathrm{Gr}(l, V)$.

Let us denote by $P(l \times k)$ the set of l-tuples of integers $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that $k \geqslant \lambda_1 \geqslant \ldots \geqslant \lambda_l \geqslant 0$. Note that $P(l \times k)$ is nothing else but the set of partitions which lie in the rectangle $l \times k$. For a flag $F_\bullet$ and $\lambda \in P(l \times k)$ we define the Schubert cell $\Omega^\circ_\lambda(F_\bullet) \subset \mathrm{Gr}(l, V)$ as follows:

$$\Omega^\circ_\lambda(F_\bullet) := \{W \in \mathrm{Gr}(l, V) \mid \dim(W \cap F_{k-i+\lambda_i}) = i \ \forall\ i = 1, \ldots, l\}.$$

Schubert cell $\Omega^\circ_\lambda(F_\bullet)$ has codimension $|\lambda|$ in $\mathrm{Gr}(l, V)$, and is isomorphic to $\mathbb{A}^{lk-|\lambda|}$. We have a disjoint decomposition

$$\mathrm{Gr}(l, V) = \bigsqcup_{\lambda \in P(l \times k)} \Omega^\circ_\lambda(F_\bullet).$$

To each $\lambda \in P(l \times k)$ we can also associate a Schubert variety which can be defined as follows:

$$\Omega_\lambda(F_\bullet) := \{W \in \mathrm{Gr}(l, V) \mid \dim(W \cap F_{k-i+\lambda_i}) \geqslant i \ \forall\ i = 1, \ldots, l\}.$$

Varieties $\Omega_\lambda(F_\bullet)$ are closed subvarieties of $\mathrm{Gr}(l, V)$ of codimension $|\lambda|$, we can also describe $\Omega_\lambda(F_\bullet)$ as the Zariski closure of $\Omega^\circ_\lambda(F_\bullet)$.

Example 1.1. *For $l = 1$ we have $\mathrm{Gr}(l, V) = \mathbb{P}^{n-1}$ and Schubert varieties are parametrized by numbers $0 \leqslant a \leqslant n-1$. Schubert variety corresponding to $0 \leqslant a \leqslant n-1$ and a flag $F_\bullet$ is precisely $\mathbb{P}(F_{n-a}) \subset \mathbb{P}(V)$.*

For $\lambda \in P(l \times k)$ we denote by $\sigma_\lambda \in H^{2|\lambda|}(\mathrm{Gr}(l, V), \mathbb{Z})$ the cohomology class of $\Omega_\lambda(F_\bullet)$ (note that it does not depend on $F_\bullet$ since for any two flags $F_\bullet, F'_\bullet$ there exists $g \in \mathrm{GL}(V)$ such that $g(F_\bullet) = F'_\bullet$ so $g(\Omega_\lambda(F_\bullet)) = \Omega_\lambda(F'_\bullet)$ and now it remains to note that $\mathrm{GL}(V)$ is connected). For $\lambda_1, \lambda_2, \ldots, \lambda_N \in P(l \times k)$ we denote by $\langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle \in \mathbb{Z}$ the intersection pairing of these subvarieties of $\mathrm{Gr}(l, V)$ (which is by the definition zero if $|\lambda_1| + \ldots + |\lambda_N| \neq \dim \mathrm{Gr}(l, V) = lk$).

For a partition $\lambda \in P(l \times k)$ we denote by λ^c the following partition: $\lambda^c = (k - \lambda_l, k - \lambda_{l-1}, \ldots, k - \alpha_1)$. The following proposition is standard.

Proposition 1.2. *For $\lambda, \mu \in P(l \times k)$ we have*

$$\langle \Omega_\lambda, \Omega_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu^c \\ 0 & \text{otherwise}. \end{cases}$$

Corollary 1.3. *For any $\lambda_1, \lambda_2, \ldots, \lambda_N \in P(l \times k)$ we have*

$$\sigma_{\lambda_1} \cdot \ldots \cdot \sigma_{\lambda_N} = \sum_{\mu} \langle \Omega_{\mu^c}, \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle \sigma_{\mu}.$$

Proof. We can write $\sigma_{\lambda_1} \cdot \ldots \cdot \sigma_{\lambda_N} = \sum_{\mu} c_{\mu}\sigma_{\mu}$ for some $c_{\mu} \in \mathbb{Z}$. It follows from proposition 1.2 that $\sigma_{\mu^c} \cdot \sigma_{\lambda_1} \cdot \ldots \cdot \sigma_{\lambda_N} = c_{\mu}\sigma_{\mu^c} \cdot \sigma_{\mu}$. It again follows from proposition 1.2 and definitions that

$$\sigma_{\mu^c} \cdot \sigma_{\lambda_1} \cdot \ldots \cdot \sigma_{\lambda_N} = \langle \Omega_{\mu^c}, \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle \sigma_{\mu^c} \cdot \sigma_{\mu}$$

and the claim follows. □

1.2 Pieri and Giambelli formulas

To $1 \geqslant p \geqslant k$ we can associate a partition $(p, 0, 0, \ldots, 0) \in P(l \times k)$ and denote by Ω_p the corresponding Schubert variety.

1.2.1 Pieri formula

Proposition 1.4. *We have $\sigma_p \cdot \sigma_{\alpha} = \sum_{\beta} \sigma_{\beta}$, where the sum is taken over all β that can be obtained by adding i boxes to α with no two in the same column.*

Remark 1.5. Note that the Pieri formula is equivalent to the following statement about intersection pairing of Schubert varieties. If $\alpha, \beta \in P(l \times k)$, $0 \leqslant p \leqslant k$ are such that $|\alpha| + |\beta| + p = \dim \operatorname{Gr}(l, V) = l(n-l)$ then

$$\langle \Omega_{\alpha}, \Omega_{\beta}, \Omega_p \rangle = \begin{cases} 1 \text{ if } \alpha_i + \beta_{l-i} \geqslant n - l \text{ and } \alpha_i + \beta_{l+1-i} \leqslant n - l \\ 0, \text{ otherwise}. \end{cases}$$

Indeed recall that

$$\sigma_p \cdot \sigma_{\alpha} = \sum_{\beta} \langle \Omega_{\alpha}, \Omega_{\beta^c}, \Omega_p \rangle \sigma_{\beta}.$$

Note now that β can be obtained by adding p boxes to α with no two in the same column iff $\beta_i \geqslant \alpha_i$ for every $i = 1, 2, \ldots, l$ and $\alpha_i \geqslant \beta_{i+1}$ for $i = 1, 2, \ldots, l-1$. . Recall now that $\beta_i^c = n - l - \beta_{l+1-i}$ i.e. $\beta_i = n - l - \beta_{l+1-i}^c$ for every $i = 1, 2, \ldots, l$. We conclude that the conditions $\beta_i \geqslant \alpha_i$, $\alpha_i \geqslant \beta_{i+1}$ are equivalent to $n - l - \beta_{l+1-i}^c \geqslant \alpha_i$, $\alpha_i \geqslant n - l - \beta_{l-i}^c$ i.e. $n - l \geqslant \alpha_i + \beta_{l+1-i}^c$, $\alpha_i + \beta_{l-i}^c \geqslant n - l$ respectively and the claim follows.

1.2.2 Giambelli formula

Let us now recall the classical Giambelli formula which allows to compute Schubert classes σ_λ in terms of Schubert classes σ_a, $0 \leqslant a \leqslant k$.

Theorem 1.6. *If λ is a partition contained in $l \times k$ rectangle then the Schubert class σ_λ in $H^*(\mathrm{Gr}(l, V), \mathbb{Z})$ is given by $\sigma_\lambda = \det(\sigma_{\lambda_i+j-i})$, where $\sigma_i = 0$ for $i < 0$ or $i > k$.*

Corollary 1.7. *Ring $H^*(\mathrm{Gr}(l, V), \mathbb{Z})$ is generated (as an algebra over $\mathbb{Z}$) by Schubert classes σ_a, $0 \leqslant a \leqslant k$.*

1.3 Representation via generators and relations

We finish this section by recalling theorem which describes the ring $H^*(\mathrm{Gr}(l, V), \mathbb{Z})$ explicitly (using generators and relations).

Theorem 1.8. *We have an isomorphism*

$$H^*(\mathrm{Gr}(l, V), \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_l, q]/(y_{k+1}, \ldots, y_{n-1}, y_n)$$

where $y_p := \det(c_{1+j-i})_{1 \leq i,j \leq p}$. Element x_i corresponds to the ith Chern class of the dual of the tautological bundle on $\mathrm{Gr}(l, V)$.

2 Moduli spaces of rational curves and quantum cohomology

We fix a flag $F_\bullet$. Recall that $X = \mathrm{Gr}(l, W)$ is covered by Schubert cells $\Omega^\circ_\lambda(F_\bullet)$, where λ runs through partitions $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that $n - l \geqslant \lambda_1 \geqslant \ldots \geqslant \lambda_l \geqslant 0$, recall that we denote the set of such partitions by $P(l \times k)$. Recall also that we define $\Omega_\lambda(F_\bullet)$ as the closure of $\Omega^\circ_\lambda(F_\bullet)$ and call it a Schubert variety corresponding to λ.

For each integer $d \geqslant 0$ and the collection of partitions $\lambda_1, \ldots, \lambda_N$ we will define the number $\langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle_d$ which can be thought as follows. Choose generic points $p_1, \ldots, p_N \in \mathbb{P}^1$ and generic flags $F^1_\bullet, \ldots, F^N_\bullet$. Then $\langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle_d$ is the number of algebraic morphisms $f : \mathbb{P}^1 \to X$ of degree d such that $f(p_i) \in \Omega_{\lambda_i}(F^i_\bullet)$ and is zero if the set of such maps is infinite.

Remark 2.1. Note that Schubert varieties $\Omega_\lambda(F_\bullet)$ differ from $\Omega_{\lambda_i}(F^i_\bullet)$ by the action of some element $g_i \in \mathrm{GL}(V)$ so the varieties $\Omega_{\lambda_i}(F^i_\bullet)$ can be thought as generic translates of the varieties $\Omega_{\lambda_i}(F_\bullet)$ respectively.

Remark 2.2. Note that for $d = 0$ the number $\langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle_0$ is just the intersection number $\langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle$. Indeed morphism $f\colon \mathbb{P}^1 \to X$ of degree d should map whole $\mathbb{P}^1$ to some point $x \in X$. Now from the conditions $f(p_i) \in \Omega_{\lambda_i}(F^i_\bullet)$ we conclude that $p \in \bigcap_i \Omega_{\lambda_i}(F^i_\bullet)$ so $\langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle_0 = \#(\bigcap_{i=1}^N \Omega_{\lambda_i}(F^i_\bullet)) = \langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle$, where the last equality holds since varieties $\Omega_{\lambda_i}(F_\bullet), \Omega_{\lambda_i}(F^i_\bullet)$ have the same cohomology classes in $H^*(\mathrm{Gr}(l, V), \mathbb{C})$ since the differ by the action of some $g_i \in \mathrm{GL}(V)$ and $\mathrm{GL}(V)$ is connected.

To give a rigorous definition of the number $\langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle_d$ for $d > 0$ we need to understand how to think about the moduli space $\mathcal{M}_d$ of morphisms of degree d from $\mathbb{P}^1$ to X geometrically. Note that for $d = 0$ this space is naturally identifies with X and by remark 2.2 we can define $\langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle_0 := \langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle$. For $d > 0$ we will analogically define $\langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle_d$ as an intersection pairing of certain varieties in a certain compactification of $\mathcal{M}_d$.

To construct a scheme structure on $\mathcal{M}_d$ we will first of all describe the functor $\mathbf{Sch}^{opp} \to \mathbf{Set}$ which it should represent and then will deduce from classical Grothendieck results that this functor is indeed represented by some smooth quasi-projective scheme of finite type. We start from recalling a description of the functor

$$\mathbf{Sch}^{opp} \to \mathbf{Set},\ S \mapsto \mathrm{Map}(S, \mathrm{Gr}(l, V))$$

which represents Grassmannian $\mathrm{Gr}(l, V)$. Let $\mathcal{U}$ be the tautological vector bundle on $\mathrm{Gr}(l, V)$ of rank l which can be defined as follows. Recall the identification $\mathrm{Gr}(l, V) \simeq \mathrm{GL}(V)/P$ and consider the standard representation $P \curvearrowright U$. Then we can form the associated vector bundle $\mathcal{U} := \mathrm{GL}(V) *_P U$ which we will call *tautological*.

Remark 2.3. Recall that if G is an algebraic group and $H \subset G$ is an algebraic subgroup then to any finite dimensional representation $H \curvearrowright W$ we can associate a vector bundle $G *_H W$ which can be defined as follows. We have the following free right action $G \times W \curvearrowleft H$, $(g, w).h = (gh, h^{-1}w)$ then $G *_H W := (G \times W)/H$. Note that $G *_H W$ has the natural projection morphism $G *_H W \to G/H$ which makes it a vector bundle.

Remark 2.4. Note that if the action $H \curvearrowright W$ can be extended to the action $G \curvearrowright W$ then the vector bundle $G *_H W$ is trivial. Indeed we have the isomorphism $(G/H) \times W \xrightarrow{\sim} G *_H W$ given by $([g], w) \mapsto (g, g^{-1}w)$.

Note that we have the natural embedding of vector bundles

$$\mathcal{U} = \mathrm{GL}(V) *_P U \hookrightarrow \mathrm{GL}(V) *_P V = V \otimes \mathcal{O}_X$$

which corresponds to the embedding $U \hookrightarrow V$. Under this embedding fiber of $\mathcal{U}$ over a point $W \in \mathrm{Gr}(l, V)$ identifies with $W \subset V$.

Vector bundle $\mathcal{U}$ also has the following description which will be useful in the proof of proposition 2.5. Recall the Plücker embedding

$$\iota\colon \mathrm{Gr}(l, V) \hookrightarrow \mathbb{P}(\Lambda^l V),\ W \mapsto \Lambda^l W.$$

Recall that we have a natural derivative (contraction) map

$$\mathrm{contr}\colon \Lambda^{l-1} V^* \otimes \Lambda^l V \to V,\ f \otimes v \mapsto \partial_f(v).$$

Then the dual map $\mathrm{contr}^*\colon V^* \to \Lambda^{k-1} V \otimes \Lambda^k V^*$ gives us the morphism

$$V^* \otimes \mathcal{O}_{\mathbb{P}(\Lambda^l V)} \to \Lambda^{l-1} V \otimes \mathcal{O}_{\mathbb{P}(\Lambda^l V)}(1)$$

induced by the isomorphism $\Gamma(\mathbb{P}(\Lambda^l V), \mathcal{O}_{\mathbb{P}(\Lambda^l V)}(1)) \simeq \Lambda^l V^*$.

By taking duals it gives us the morphism

$$\Phi\colon \Lambda^{l-1} V^* \otimes \mathcal{O}_{\mathbb{P}(\Lambda^l V)}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}(\Lambda^l V)}.$$

Fiberwise this morphism can be described as follows. Note that $\mathcal{O}_{\mathbb{P}(\Lambda^l V)}(-1)$ is a tautological bundle on $\mathbb{P}(\Lambda^l V)$ so its fiber over a point $P \in \mathbb{P}(\Lambda^l V)$ is $P \subset \Lambda^l V$ considered as 1-dimensional vector space. Now starting from a vector $v \in P \subset \Lambda^l V$ and $f \in \Lambda^{l-1} V^*$ have $\Phi_P(f \otimes v) = \partial_f(v)$. It now follows from the definitions that we have $\mathcal{U} = \mathrm{Im}\, \Phi|_{\mathrm{Gr}(l,V)}$ since for any vector $v \in \Lambda^l V$ the support $\mathrm{Supp}(v) \subset V$ of this vector coincides with the image $\mathrm{contr}(\Lambda^{l-1} V^* \otimes \mathbb{C}v)$.

We are now ready to formulate and prove the universal property of $\mathrm{Gr}(l, V)$.

Proposition 2.5. *For $S \in$ **Sch** the set $\mathrm{Map}(S, \mathrm{Gr}(l, V))$ identifies with the set of pairs $(\varphi, \mathcal{E})$ consisting of a vector bundle $\mathcal{E}$ of rank l on S and an injection of vector bundles $\mathcal{E} \hookrightarrow V \otimes \mathcal{O}_S$.*

Proof. If $f\colon S \to X$ is any morphism then we can set $\mathcal{E} := f^*(\mathcal{U})$ and $\varphi := f^*(\mathcal{U} \hookrightarrow V \otimes \mathcal{O}_X)$.

In the opposite direction starting from a pair $(\varphi, \mathcal{E})$ we can consider a morphism

$$\Lambda^l\varphi\colon \Lambda^l\mathcal{E} \to \Lambda^l V \otimes \mathcal{O}_S.$$

It follows from the universal property of $\mathbb{P}(\Lambda^l V)$ that there exists a morphism $\bar{f}\colon S \to \mathbb{P}(\Lambda^l V)$ such that $\bar{f}^*(\mathcal{O}_{\mathbb{P}(\Lambda^l V)}(-1)) \simeq \Lambda^l\mathcal{E}$. It is clear from the definitions that the image of $\bar{f}$ lies in $\mathrm{Gr}(l, V) \subset \mathbb{P}(\Lambda^l V)$ so we obtain the desired morphism $f\colon S \to \mathrm{Gr}(l, V)$. □

We conclude that a morphism $\mathbb{P}^1 \to \mathrm{Gr}(l, V)$ is the same as the pair of rank l vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ and an embedding $\mathcal{E} \hookrightarrow V \otimes \mathcal{O}_{\mathbb{P}^1}$. Note that the embedding of vector bundles $\mathcal{E} \hookrightarrow V \otimes \mathcal{O}_{\mathbb{P}^1}$ is the same as the surjection of vector bundles $V^* \otimes \mathcal{O}_{\mathbb{P}^1} \twoheadrightarrow \mathcal{E}^*$. Note also that the condition that f has degree d precisely means that $\deg(\mathcal{E}^*) = d$, here $\deg(\mathcal{E}^*) := \deg \Lambda^{top}\mathcal{E}^*$.

Let us now define the functor $\mathcal{M}_d\colon \mathbf{Sch}^{opp} \to \mathbf{Set}$. Note that the set $\mathcal{M}_d(\mathrm{Spec}(\mathbb{C})) =: \mathcal{M}_d(\mathbb{C})$ must parametrize morphisms $f\colon \mathbb{P}^1 \to X$ of degree d or equivalently (by proposition 2.5) pairs $(\pi, \mathcal{E})$ consisting of a vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ of rank l and a surjection $V \otimes \mathcal{O}_{\mathbb{P}^1} \twoheadrightarrow \mathcal{E}$ of vector bundles such that $\deg(\mathcal{E}) = d$.

Recall that if $\mathcal{F}$ is a coherent sheaf on a projective scheme X with a fixed embedding $\iota\colon X \hookrightarrow \mathbb{P}^m$ then there exists a unique polynomial $P_{\mathcal{F}}(t)$ of degree $\leqslant m$ such that $P_{\mathcal{F}}(n) = \dim_{\mathbb{C}}(\Gamma(X, \mathcal{F}(n)))$ for $n \in \mathbb{Z}, n \gg 0$. This polynomial is called Hilbert polynomial of (X, ι).

Example 2.6. *For $X = \mathbb{P}^1$, $\mathcal{F}$ a vector bundle of rank l and degree d, $\iota = \mathrm{Id}_{\mathbb{P}^1}$ we have $P_{\mathcal{F}}(t) = tl + l + d$. Indeed by Birkhoff–Grothendieck theorem every such $\mathcal{F}$ is isomorphic to the direct sum $\mathcal{O}(m_1) \oplus \ldots \oplus \mathcal{O}(m_l)$ for some $m_i \in \mathbb{Z}$ so for $m \gg 0$ we have $\dim_{\mathbb{C}} \Gamma(\mathbb{P}^1, \mathcal{F}(m)) = (m_1 + m + 1) + \ldots + (m_l + m + 1) = lm + l + d$.*

So we see that the condition that $\mathcal{E} \in \mathrm{Vect}(\mathbb{P}^1)$ has degree d and rank l can be compactly rewritten as $P_{\mathcal{E}}(t) = tl + l + d$. Note also that this approach allows us to associate rank and degree to any coherent sheaf $\mathcal{E} \in \mathrm{Coh}(\mathbb{P}^1)$. Indeed if $P_{\mathcal{E}}(t) = at + b$ then we set $r(E) := a$ and $d(\mathcal{E}) := b - a$ and call them rank and degree respectively.

Definition 2.7. Pick a test scheme $S \in \mathbf{Sch}$ then $\mathcal{M}_d(S)$ is the set of pairs $(\mathcal{E}, \pi)$, where $\mathcal{E}$ is a locally free sheaf (vector bundle) on $\mathbb{P}^1 \times S$ flat over S such that $P_{\mathcal{E}|_{\mathbb{P}^1\times s}}(t) = tl+l+d$ for any geometric point s of S, $\pi\colon V^*\otimes\mathcal{O}_{\mathbb{P}^1\times S}\twoheadrightarrow\mathcal{E}$ is the surjection of *vector bundles*.

Theorem 2.8. *The functor $\mathcal{M}_d$ is represented by a smooth quasi-projective scheme of dimension $nd+(n-l)l$ to be denoted by the same symbol.*

To prove this theorem we define a smooth projective variety $\mathcal{Q}_d$ together with an open embedding $\mathcal{M}_d\hookrightarrow\mathcal{Q}_d$.

Definition 2.9. Pick a test scheme $S\in\mathbf{Sch}$ then $\mathcal{Q}_d(S)$ is the set of pairs $(\mathcal{E},\pi)$, where $\mathcal{E}$ is a coherent sheaf on $\mathbb{P}^1\times S$ flat over S and such that $P_{\mathcal{E}|_{\mathbb{P}^1\times s}}(t) = tl+l+d$ for any geometric point s of S, $\pi\colon V^*\otimes\mathcal{O}_{\mathbb{P}^1\times S}\to\mathcal{E}$ is the surjection of *sheaves*.

Note that we have the natural embedding of functors $\mathcal{M}_d\hookrightarrow\mathcal{Q}_d$. Now theorem 2.8 follows from the theorem bellow.

Theorem 2.10. *Functor $\mathcal{Q}_d$ is represented by a smooth projective scheme of dimension $nd+(n-l)l$ and the morphism $\mathcal{M}_d\hookrightarrow\mathcal{Q}_d$ identifies $\mathcal{M}_d$ with an open subscheme of $\mathcal{Q}_d$.*

Proof. Follows from [Gr], see also [N]. $\square$

Example 2.11. *For $l=1$ (i.e. $\mathrm{Gr}(l,V)=\mathbb{P}^{n-1}$) we have $\mathcal{Q}_d=\mathbb{P}^{n(d+1)-1}$. It can be easily seen at the level of $\mathbb{C}$-points: a point of $\mathcal{Q}_d(\mathrm{Spec}\,\mathbb{C})$ consists of a coherent sheaf $\mathcal{E}\in\mathrm{Coh}(\mathbb{P}^1)$ of degree d and rank 1 and a surjection of sheaves $\pi\colon V^*\otimes\mathcal{O}_{\mathbb{P}^1}\to\mathcal{E}$. Sheaf $\mathcal{E}$ must be of the form $\mathcal{O}(d)\oplus\mathcal{F}$, where $\mathcal{F}$ is the sum of finite number of skyscraper sheaves on $\mathbb{P}^1$. There are no nonzero morphisms from $\mathcal{O}_{\mathbb{P}^1}$ to any skyscraper sheaf so we conclude that $\mathcal{F}=0$ (otherwise there are no surjection $V^*\otimes\mathcal{O}_{\mathbb{P}^1}\to\mathcal{E}$) i.e. $\mathcal{E}=\mathcal{O}(d)$. We see that*

$$\mathrm{Hom}(V^*\otimes\mathcal{O}_{\mathbb{P}^1},\mathcal{E})=\mathrm{Hom}(V^*\otimes\mathcal{O}_{\mathbb{P}^1},\mathcal{O}(d))\simeq V\otimes S^d(\mathbb{C}^{2^*})$$

and an element $f\in\mathrm{Hom}(V^*\otimes\mathcal{O}_{\mathbb{P}^1},\mathcal{E})$ defines a surjective morphism of sheaves iff $f\neq 0$. We conclude that $\mathcal{Q}_d=\mathbb{P}(V\otimes S^d(\mathbb{C}^{2^*}))=\mathbb{P}^{n(d+1)-1}$.

Note that by the universal property of $\mathcal{Q}_d$ applied to $S=\mathcal{Q}_d$ and $\mathrm{Id}\colon\mathcal{Q}_d\to\mathcal{Q}_d$ we obtain a universal exact sequence of sheaves on $\mathbb{P}^1\times\mathcal{Q}_d$:

$$0\to\mathcal{S}_d\to V^*\otimes\mathcal{O}_{\mathbb{P}^1\times\mathcal{Q}_d}\to\mathcal{T}_d\to 0.$$

Sheaf $\mathcal{S}_d$ is locally free (follows from the fact that $\mathcal{T}_d$ is flat over $\mathcal{Q}_d$). So we can consider the dual universal map

$$u\colon V \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathcal{Q}_d} \to \mathcal{S}_d^*.$$

Now we can think about moduli space of morphisms $\mathbb{P}^1 \to \mathrm{Gr}(l, V)$ as about a smooth algebraic variety $\mathcal{M}_d$ and we want to define algebraic varieties $\Omega_{\lambda,p}(F_\bullet) \subset \mathcal{M}_d$ which would parametrize morphisms $f\colon \mathbb{P}^1 \to \mathcal{M}_d$ such that $f(p) \in \Omega_\lambda(F_\bullet)$, here $\lambda \in P(l \times k)$ and $p \in \mathbb{P}^1$ is some point. To do so we need to define an evaluation morphism $\mathrm{ev}\colon \mathbb{P}^1 \times \mathcal{M}_d \to \mathrm{Gr}(l, V)$.

Definition 2.12. For $S \in \mathbf{Sch}$ define a map $\mathrm{ev}(S)\colon (\mathbb{P}^1 \times \mathcal{M}_d)(S) \to \mathrm{Gr}(l, V)(S)$ as follows: note that a morphism $f\colon S \to \mathbb{P}^1 \times \mathcal{M}_d$ defines us a morphism $S \to \mathcal{M}_d$ (via the projection $\mathbb{P}^1 \times \mathcal{M}_d \to \mathcal{M}_d$) which is the same as the pair $(\mathcal{E}, \pi)$ as in definition 2.7. Consider now the natural embedding $\iota_p\colon p \times S \hookrightarrow \mathbb{P}^1 \times S$ and send f to $(\iota_p^*\mathcal{E}, \iota_p^*\pi) \in \mathrm{Gr}(l, V)(S)$.

Note that directly from the definitions ev is a morphism of functors so it induces a morphism of the corresponding varieties to be denoted by the same symbol.

Example 2.13. *For $l = 1$ we have $\mathrm{Gr}(l, V) = \mathbb{P}(V)$ and recall that $\mathcal{Q}_d = \mathbb{P}(V \otimes S^d\mathbb{C}^{2^*})$ and $\mathcal{M}_d \subset \mathbb{P}(V \otimes S^d\mathbb{C}^{2^*})$ is an open subset. Then the morphism $\mathrm{ev}\colon \mathbb{P}^1 \times \mathcal{M}_d \to \mathbb{P}(V)$ is a restriction to $\mathbb{P}^1 \times \mathcal{M}_d$ of the* birational *morphism*

$$\mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(V \otimes S^d\mathbb{C}^{2^*}) \to \mathbb{P}(V)$$

induced by the map $\mathbb{C}^2 \otimes (V \otimes S^d\mathbb{C}^{2^*}) \to V$, $x \otimes v \otimes f \mapsto f(x)v$.

Definition 2.14. For $p \in \mathbb{P}^1$, $\lambda \in P(l \times k)$ and a flag $F_\bullet$ we define $\Omega_{\lambda,p}(F_\bullet)$ as the intersection $\mathrm{ev}^{-1}(\Omega_\lambda(F_\bullet)) \cap (p \times \mathcal{M}_d)$. It can be considered as a subscheme of $\mathcal{M}_d$ of codimension $|\lambda|$.

Example 2.15. *Assume that $l = 1$. Recall that for a fixed flag $F_\bullet$ and $0 \leqslant a \leqslant n-1$ we have $\Omega_a(F_\bullet) = \mathbb{P}(F_{n-a}) \subset \mathbb{P}(V)$. It follows from the example 2.13 that for a point $p = [x : y] \in \mathbb{P}^1$ we have $\Omega_{a,p}(F_\bullet) = \mathbb{P}(\tilde{F}_{n-a}) \cap \mathcal{M}_d$, where the intersection is taken in $\mathbb{P}(V \otimes S^d\mathbb{C}^{2^*})$ and $\tilde{F}_{n-a}$ is the preimage of F_{n-a} under the linear map*

$$V \otimes S^d\mathbb{C}^{2^*} \to V,\ v \otimes f \mapsto f(x, y)v.$$

Let us now define compactifications $\overline{\Omega}_{\lambda,p}(F_\bullet)$ of the varieties $\Omega_{\lambda,p}(F_\bullet)$ in $\mathcal{Q}_d$ (we will then define the desired numbers $\langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle_d$ as intersection pairings of cohomology classes $\overline{\Omega}_{\lambda,p}(F_\bullet)$ in the smooth variety $\mathcal{Q}_d$). For $l = 1$ we will just have $\overline{\Omega}_{a,p}(F_\bullet) = \mathbb{P}(\tilde{F}_{n-a}) \subset \mathcal{Q}_d$ (see the example 2.15 for notations).

As we already see in the case $l = 1$ (see example 2.13) the morphism $\mathrm{ev}\colon \mathbb{P}^1 \times \mathcal{M}_d \to \mathrm{Gr}(l, V)$ does not extend to a morphism $\mathbb{P}^1 \times \mathcal{Q}_d \to \mathrm{Gr}(l, V)$ so we can not define varieties $\overline{\Omega}_{\lambda,p}(F_\bullet) \subset \mathcal{Q}_d$ directly in the same way as we have defined varieties $\Omega_{\lambda,p}(F_\bullet) \subset \mathcal{M}_d$ in definition 2.14. We will use a universal morphism $u\colon V \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathcal{Q}_d} \to \mathcal{S}_d^*$ to define them.

Definition 2.16. For each $i = 1, \ldots, l$, let $D_{i,\lambda_i}(F_\bullet) \subset \mathbb{P}^1 \times \mathcal{Q}_d$ be the largest subscheme on which the dimension of the kernel of $u\colon F_{n-l-i+\lambda_i} \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathcal{Q}_d} \to \mathcal{S}_d^*$ is at least i, and let $D_{i,\lambda_i,p}(F_\bullet)$ be the intersection $D_{i,\lambda_i,p}(F_\bullet) \cap (p \times \mathcal{Q}_d)$ considered as a subscheme of $\mathcal{Q}_d$. Then we define

$$\overline{\Omega}_{\lambda,p}(F_\bullet) := D_{1,\lambda_1,p}(F_\bullet) \cap \ldots \cap D_{l,\lambda_l,p}(F_\bullet).$$

This is a subscheme of $\mathcal{Q}_d$ of codimension $|\lambda|$.

It is clear from the definitions that $\overline{\Omega}_{\lambda,p}(F_\bullet) \cap \mathcal{M}_d = \Omega_{\lambda,p}(F_\bullet)$.

Example 2.17. *It follows from example 2.15 and the definitions that for* $l = 1$ *and* $0 \leqslant a \leqslant n-1$ *we have* $\overline{\Omega}_{a,p}(F_\bullet) = \mathbb{P}(\tilde{F}_{n-a})$. *Note also that*

$$\dim \overline{\Omega}_{a,p}(F_\bullet) = \dim(\tilde{F}_{n-a}) - 1 = nd + \dim(F_{n-a}) = nd + n - a - 1$$

since the map

$$V \otimes S^d\mathbb{C}^{2*} \to V,\ v \otimes f \mapsto f(x,y)v$$

is clearly surjective. We conclude that the codimension of $\overline{\Omega}_{a,p}(F_\bullet)$ *in* $\mathcal{Q}_d$ *indeed equals to* a.

Let us denote by $\overline{\sigma}_\lambda \in H^{|\lambda|}(\mathcal{Q}_d, \mathbb{Z})$ the cohomology class of $\overline{\Omega}_{\lambda,p}(F_\bullet)$. We are now ready to define Gromov-Witten numbers $\langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle_d$.

Definition 2.18. For $\lambda_1, \ldots, \lambda_N \in P(l \times k)$ let $\langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle_d$ be zero if $|\lambda_1| + |\lambda_2| + \ldots + |\lambda_l| \neq \dim \mathcal{M}_d = nd + (n-l)l$. Otherwise we define $\langle \Omega_{\lambda_1}, \ldots, \Omega_{\lambda_N} \rangle_d$ as the intersection pairing of the cohomology classes $\overline{\sigma}_{\lambda_i} \in H^*(\mathcal{Q}_d, \mathbb{C})$.

Example 2.19. *Assume $l = 1$ then $\lambda_1, \ldots, \lambda_N$ are just some numbers $a_1, \ldots, a_N$ such that $0 \leqslant a_i \leqslant n-1$. Let us now compute the numbers $\langle \Omega_{a_1}, \ldots, \Omega_{a_N} \rangle_d$. We assume that $a_1 + \ldots + a_N = n(d+1) - 1$ (otherwise this number is zero by the definition). It follows from example 2.11 that $\mathcal{Q}_d = \mathbb{P}(V \otimes S^d \mathbb{C}^{2*}) = \mathbb{P}^{n(d+1)-1}$ so we have an isomorphism $H^*(\mathcal{Q}_d, \mathbb{Z}) \simeq \mathbb{Z}[t]/t^{n(d+1)}$ and by example 2.17 we have $\overline{\Omega}_{p,a}(F_\bullet) = \mathbb{P}(\tilde{F}_{n-a_i})$ for certain subspace $\tilde{F}_{n-a_i} \subset V \otimes S^d \mathbb{C}^{2*}$ of codimension a_i so we have $\overline{\sigma}_{a_i} = t^{a_i}$. We conclude that $\langle \Omega_{a_1}, \ldots, \Omega_{a_N} \rangle_d = 1$.*

2.1 Small quantum cohomology ring

We can now define the small quantum cohomology ring $QH^*(\mathrm{Gr}(l,V),\mathbb{Z}) := H^*(\mathrm{Gr}(l,V),\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ and set $\tilde{\sigma}_\lambda := \sigma_\lambda \otimes 1$. The ring structure on $QH^*(\mathrm{Gr}(l,V),\mathbb{Z})$ is defined by

$$\tilde{\sigma}_\lambda \cdot \tilde{\sigma}_\mu := \sum_{\nu,\, d \geqslant 0} \langle \Omega_\lambda, \Omega_\mu, \Omega_{\nu^t} \rangle_d q^d \tilde{\sigma}_\nu.$$

It is a nontrivial fact that $\cdot$ defines an associative ring structure.

Example 2.20. *Consider the case $l = 1$. In this case λ, μ, ν are just numbers $0 \leqslant a, b, c \leqslant n-1$ and it follows from example 2.19 that $\langle \Omega_a, \Omega_b, \Omega_{n-1-c} \rangle_d = 0$ if $a+b+n-c \neq n(d+1)$ and is 1 otherwise. We conculde that $QH^*(\mathrm{Gr}(l,V),\mathbb{Z})$ is isomorphic to $\mathbb{Z}[t,q]/(t^n - q)$ via the map $\sigma_a \mapsto t^a$. Note that for $q = 0$ we obtain the cohomology ring of $\mathbb{P}^{n-1}$.*

In the example bellow we have explicitly described the ring $QH^*(\mathrm{Gr}(l,V),\mathbb{Z})$ for $l = 1$. The goal of the next sections is to generalize this result to the case of arbitrary l.

3 Main tools

We start from the following definition.

Definition 3.1. If A is a subset of $\mathrm{Gr}(l,V)$ then we define $\operatorname{Span} A$ to be the sum $\sum_{W \in A} W$. We also define $\ker A$ as the intersection $\bigcap_{W \in A} W$.

Example 3.2. *For $A = \{W\} \in \mathrm{Gr}(l,V)$ we have $\operatorname{Span} A = \ker A = W \subset V$. For $A = \mathrm{Gr}(l,V)$ we have $\operatorname{Span} A = V$, $\ker A = 0$.*

Remark 3.3. Note that for a fixed subspace $F \subset V$ and a subvariety $A \subset \operatorname{Gr}(l, V)$ we have $\operatorname{Span} A \subset F$ (resp. $F \subset \ker A$) iff $A \subset \operatorname{Gr}(l, K)$ (resp. $A \subset \operatorname{Gr}(l - \dim K, V/K) \subset \operatorname{Gr}(l, V)$). Note also that if $i\colon \operatorname{Gr}(l, V) \xrightarrow{\sim} \operatorname{Gr}(n - l, V^*)$, $W \mapsto \operatorname{Ann} W$ is the natural identification. Then

$$\operatorname{Span} i(A) = \operatorname{Ann}(\ker A),\ \ker A = \operatorname{Ann}(\operatorname{Span} A).$$

Lemma 3.4. *Let C be a rational curve of degree d in X. Then* $\operatorname{Span} C$ *has dimension at most $l + d$ and* $\ker C$ *has dimension at least $l - d$.*

Proof. Curve C is an image of a regular function $f\colon \mathbb{P}^1 \to X$ of degree d. By proposition 2.5 this map corresponds to $\mathcal{E} = f^*(\mathcal{U}) \subset V \otimes \mathcal{O}_{\mathbb{P}^1}$, $\varphi := f^*(\mathcal{U} \hookrightarrow V \otimes \mathcal{O}_{\mathbb{P}^1})$. Point $p \in \mathbb{P}^1$ goes to $\mathcal{E}_p \subset V \in X$. Condition that $\deg f = d$ corresponds to $\deg \mathcal{E} = -d$. Therefore $\mathcal{E} = \oplus_{i=1}^l \mathcal{O}_{\mathbb{P}^1}(-a_i)$, where $a_i \geqslant 0$, $\sum_{i=1}^l a_i = d$. Each map $V^* \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(a)$ is defined by map on global sections $\phi\colon V^* \to \Gamma(\mathcal{O}_{\mathbb{P}^1}(a))$. Taking duals we see that map $\mathcal{O}_{\mathbb{P}^1}(-a) \to V \otimes \mathcal{O}_{\mathbb{P}^1}$ is given by map $\phi^*\colon \Gamma(\mathcal{O}_{\mathbb{P}^1}(a))^* \to V$. Hence map from $\mathcal{E}$ to $V \otimes \mathcal{O}_{\mathbb{P}^1}$ is given by $\phi_1^*, \ldots, \phi_l^*$. It is easy to see that for any $p \in \mathbb{P}^1$ its image is $L = \operatorname{Span}(v_1, \ldots, v_l)$ where $v_i \in \operatorname{Im} \phi_i^*$. Therefore span of C is contained in $\cup \operatorname{Im} \phi_i^*$, so it has dimension at most

$$\sum \dim \phi_i^* \leqslant \sum \dim \Gamma(\mathcal{O}_{\mathbb{P}^1}(a_i)) = \sum (1 + a_i) = l + d$$

On the other hand at least $l - d$ of a_i equal to 0. In this case $a_i = 0$ we have $\operatorname{Im} \phi_i^* = 1$. So any L contains $\operatorname{Im} \phi_i^*$ for this i. We deduce that kernel of C has dimension at least $l - d$. □

If λ is a partition and d is a nonnegative integer we define $\hat{\lambda}$ to be the partition obtained by removing the leftmost d columns from the Young diagram of d, i.e. $\hat{\lambda}_i = \max(\lambda_i - d, 0)$.

Lemma 3.5. *Let $C \subset X$ be a rational curve of degree $d \leqslant k$ and let $E \subset V$ be an $l + d$ dimensional subspace containing the span of C. If λ is a partition such that $C \cap \Omega_\lambda(F_\bullet) \neq \varnothing$ then W belongs to the Schubert variety $\Omega_{\hat{\lambda}}(F_\bullet)$ in* $\operatorname{Gr}(l + d, V)$.

Proof. Recall that the Schubert variety $\Omega_\lambda(F_\bullet)$ is defined as

$$\{W \in X \mid \dim(W \cap F_{k+i-\lambda_i}) \geqslant i\ \forall\ 1 \leqslant i \leqslant l\}.$$

Let $W \in C \cap \Omega_\lambda(F_\bullet)$. By definition $W \subset E$. Hence $\dim(E \cap F_{k+i-\lambda_i}) \geqslant \dim(W \cap F_{k+i-\lambda_i}) \geqslant i$ for all $1 \leqslant i \leqslant l$. On the other hand $F_{k-d+i-\hat{\lambda}_i} = F_{\min(k+i-\lambda_i, k-d+i)}$. Intersection of $l+d$ dimensional subspace E with $k-d+i$ dimensional subspace F_{k-d+i} has dimension at least i. Therefore $\dim(E \cap F_{k-d+i-\hat{\lambda}_i}) \geqslant i$. We deduce that E belongs to $\Omega_{\hat{\lambda}}(F_\bullet)$. $\square$

Corollary 3.6. *If $\Omega_{\hat{\lambda}} \cap \Omega_{\hat{\mu}} \cap \Omega_{\hat{\nu}} = \varnothing$ then $\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d = 0$.*

This corollary allows us to deduce statements about the quantum cohomology from statements about the usual cohomology.

Recall that for a partition λ we denote by $\ell(\lambda)$ the number of nonzero rows of λ. The following lemma is very useful.

Lemma 3.7. *Let λ and μ be partitions contained in $l \times k$ rectangle such that $l(\lambda) + l(\mu) \leqslant l$. Then*

$$\tilde{\sigma}_\lambda \cdot \tilde{\sigma}_\mu = (\sigma_\lambda \cdot \sigma_\mu) \otimes 1$$

Proof. Suppose that $d \geqslant 1$ and ν is a partition such that $|\lambda| + |\mu| + |\nu| = lk + nd$. Then we have

$$|\hat{\lambda}| + |\hat{\mu}| + |\hat{\nu}| \geqslant |\lambda| + |\mu| + |\nu| - 2ld = lk + nd - 2ld = lk + kd - ld > (l+d)(k-d)$$

It follows that for general flags $F_\bullet$, $G_\bullet$, $H_\bullet$ we have $\Omega_{\hat{\lambda}}(F_\bullet) \cap \Omega_{\hat{\mu}}(G_\bullet) \cap \Omega_{\hat{\nu}}(H_\bullet) = \varnothing$. Using Corollary 3.6 we get $\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d = 0$. The lemma follows. $\square$

4 Quantum Pieri and Giambelli formulas

Using the results of section 3 we are ready to formulate and prove quantum versions of Pieri and Giambelli formulas.

4.1 Quantum Pieri

We start from the following lemma. Recall that the number $\langle \Omega_\alpha, \Omega_\beta, \Omega_i \rangle_d$ is nonzero only if $|\alpha| + |\beta| + p = l(n-l) + dn$.

Lemma 4.1. *For $d \geqslant 1$ let $\alpha, \beta \in P(l \times k), 1 \leqslant p \leqslant n-l$ be such that $|\alpha| + |\beta| + p = l(n-l) + dn$. Then we have*

$$\langle \Omega_\alpha, \Omega_\beta, \Omega_p \rangle_d = \begin{cases} \langle \Omega_{\hat{\alpha}}, \Omega_{\hat{\beta}}, \Omega_{\hat{p}} \rangle_0 & \text{for } d = 1 \text{ and } \ell(\alpha) = \ell(\beta) = l, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Let C be a rational curve of degree d in $\mathrm{Gr}(l,V)$ which intersects with each of the varieties $\Omega_\alpha(F_\bullet)$, $\Omega_\beta(G_\bullet)$, $\Omega_p(H_\bullet)$ for generic flags $F_\bullet, G_\bullet, H_\bullet$. It follows from lemma 3.4 that there exists $E \subset V$ of dimension $l+d$ such that $\mathrm{Span}\, C \subset E$. By lemma 3.5 we must have $E \in \Omega_{\hat\alpha}(F_\bullet) \cap \Omega_{\hat\beta}(G_\bullet) \cap \Omega_{\hat p}(H_\bullet)$ and in particular $\Omega_{\hat\alpha}(F_\bullet) \cap \Omega_{\hat\beta}(G_\bullet) \cap \Omega_{\hat p}(H_\bullet) \neq \varnothing$. Recall that the codimensions of $\Omega_{\hat\alpha}(F_\bullet), \Omega_{\hat\beta}(G_\bullet), \Omega_{\hat p}(H_\bullet) \subset \mathrm{Gr}(l+d,V)$ are $|\hat\alpha|, |\hat\beta|, |\hat p|$ respectively and $\dim \mathrm{Gr}(l+d,E) = (l+d)(n-l-d)$. We conclude that

$$|\hat\alpha| + |\hat\beta| + \hat p \leqslant (l+d)(n-l-d). \tag{1}$$

Note also that α, β lie in an $l \times (n-l)$ rectangle so directly from the definitions we have $|\hat\alpha| \geqslant |\alpha| - ld$, $|\hat\beta| \geqslant |\beta| - ld$. We also have $\hat p = \max(p-d, 0) \geqslant p-d$. Altogether we obtain

$$|\hat\alpha| + |\hat\beta| + \hat p \geqslant |\alpha| + |\beta| - 2ld + p - d = (l+d)(n-l-d) + d^2 - d. \tag{2}$$

We conclude from (1) and (2) that $0 \geqslant d^2 - d$, hence, we must have $d = 1$ and moreover $\ell(\alpha) = \ell(\beta) = l$, $|\hat\alpha| + |\hat\beta| + \hat p = (l+1)(n-l-1)$. So we have shown that $\langle \Omega_\alpha, \Omega_\beta, \Omega_i \rangle_d = 0$ for $d > 1$ or if $\ell(\alpha) \neq l$ or $\ell(\beta) \neq l$. Assume that $d = 1$ and note that $\langle \Omega_{\hat\alpha}, \Omega_{\hat\beta}, \Omega_{\hat p} \rangle_0$ is nonzero only if $|\hat\alpha| + |\hat\beta| + \hat p = (l+d)(n-l-d)$. We now conclude from (2) that $\langle \Omega_{\hat\alpha}, \Omega_{\hat\beta}, \Omega_{\hat p} \rangle_0 = 0$ if $\ell(\alpha) \neq l$ or $\ell(\beta) \neq 0$.

Let us now assume that $d = 1$ and $\ell(\alpha) = \ell(\beta) = l$, hence, $\hat\alpha_i = \alpha_i - 1$, $\hat\beta_i = \beta_i - 1$ for all $i = 1, 2, \ldots, l$. Recall that our goal is to show that $\langle \Omega_\alpha, \Omega_\beta, \Omega_p \rangle_1 = \langle \Omega_{\hat\alpha}, \Omega_{\hat\beta}, \Omega_{\hat p} \rangle_0$. Recall that by proposition 1.4 we have either $\langle \Omega_{\hat\alpha}, \Omega_{\hat\beta}, \Omega_{\hat i} \rangle_0 = 0$ or $\langle \Omega_{\hat\alpha}, \Omega_{\hat\beta}, \Omega_{\hat i} \rangle_0 = 1$.

Case 1: If $\langle \Omega_{\hat\alpha}, \Omega_{\hat\beta}, \Omega_{\hat p} \rangle_0 = 0$ then we must have $\langle \Omega_\alpha, \Omega_\beta, \Omega_i \rangle_1 = 0$ since otherwise there exists a curve C of degree 1 in $\mathrm{Gr}(l,V)$ which intersects with each of the varieties $\Omega_\alpha(F_\bullet)$, $\Omega_\beta(G_\bullet)$, $\Omega_i(H_\bullet)$ for generic flags $F_\bullet, G_\bullet, H_\bullet$. It then follows from lemma 3.5 that $\mathrm{Span}\, C \subset \Omega_{\hat\alpha} \cap \Omega_{\hat\beta} \cap \Omega_{\hat i}$ and this contradicts to the fact that $\langle \Omega_{\hat\alpha}, \Omega_{\hat\beta}, \Omega_{\hat i} \rangle_0 = 0$.

Case 2: If $\langle \Omega_{\hat\alpha}, \Omega_{\hat\beta}, \Omega_{\hat i} \rangle_0 = 1$ then there exists a unique $W \subset V$ of dimension $l+1$ such that $W \in \Omega_{\hat\alpha}(F_\bullet) \cap \Omega_{\hat\beta}(G_\bullet) \cap \Omega_{\hat i}(H_\bullet)$. If C is a rational curve of degree 1 in $\mathrm{Gr}(l,V)$ wich intersects with each of the varieties $\Omega_\alpha(F_\bullet)$, $\Omega_\beta(G_\bullet)$, $\Omega_i(H_\bullet)$ then by lemma 3.5 we must have $\mathrm{Span}\, C = W$ so

$$C \subset \mathrm{Gr}(l,W) \subset \mathrm{Gr}(l,V). \tag{3}$$

Recall now that $W \in \Omega_{\hat\alpha}(F_\bullet) \cap \Omega_{\hat\beta}(G_\bullet) \cap \Omega_{\hat i}(H_\bullet)$ and flags $F_\bullet, G_\bullet, H_\bullet$ are generic so W must lie in the interiors of Schubert varieties above i.e.

$$\dim(W \cap F_{n-l-1+i-\hat\alpha_i}) = \dim(W \cap G_{n-l-1+i-\hat\beta_i}) = i.$$

Recall that $\hat{\alpha}_i = \alpha_i - 1$, $\hat{\beta}_i = \beta_i - 1$ so we conclude that

$$\dim(W \cap F_{k+i-\alpha_i}) = \dim(W \cap F_{k+i-\beta_i}) = i \ \forall\ i = 1, 2, \ldots, l. \tag{4}$$

In particular we obtain $\dim(W \cap F_{n-\alpha_l}) = \dim(W \cap G_{n-\beta_l}) = l$. Set $V_1 := W \cap F_{n-\alpha_l}$, $V_2 := W \cap G_{n-\beta_l}$. It follows from (4) that $V_1 \in \Omega_\alpha(F_\bullet)$, $V_2 \in \Omega_\beta(G_\bullet)$. Note also that codimensions of $\Omega_\alpha(F_\bullet)$, $\Omega_\beta(G_\bullet)$ in $\mathrm{Gr}(l, V)$ are $|\alpha|, |\beta|$ respectively and $|\alpha| + |\beta| = \dim \mathrm{Gr}(l, V) + n - p > \dim \mathrm{Gr}(l, V)$ so we must have $\Omega_\alpha \cap \Omega_\beta = \varnothing$ for generic $F_\bullet, G_\bullet$. It follows that $V_1 \neq V_2$ so $\dim(V_1 \cap V_2) \leqslant l - 1$. Recall also that $V_1, V_2 \subset W$ and $\dim W = l + 1$ so we must have $\dim(V_1 \cap V_2) \geqslant l - 1$. We conclude that $S := V_1 \cap V_2$ has dimension $l - 1$.

Let us return now to our C. Pick $V_1' \in C \cap \Omega_\alpha(F_\bullet)$, $V_2' \in C \cap \Omega_\beta(G_\bullet)$. By the definitions we have $V_1', V_2' \subset \mathrm{Span}\, C = W$. On the other hand by the definitions $V_1 \subset F_{n-\alpha_l}$, $V_2' \subset G_{n-\beta_l}$. We conclude that $V_1' \subset W \cap F_{n-\alpha_l}$, $V_2' \subset W \cap G_{n-\beta_l}$ so we must have $V_1' = V_1$, $V_2 = V_2'$ because of the dimension estimates. It follows that $S \subset \ker C$ but both these varieties have dimension $l - 1$ so we conclude that $S = \ker C$. It now follows from the equalities $S = \ker C$, $W = \mathrm{Span}\, C$ that $C \subset \mathbb{P}(W/S)$, hence, $C = \mathbb{P}(W/S)$ since C is projective of dimension 1 and $\mathbb{P}(W/S) \simeq \mathbb{P}^1$. So we have shown that $\langle \Omega_\alpha, \Omega_\beta, \Omega_p \rangle_1 \leqslant 1$. To show that $\langle \Omega_\alpha, \Omega_\beta, \Omega_p \rangle_1 = 1$ It remains to check that $\mathbb{P}(W/S) \subset \mathrm{Gr}(l, V)$ intersects with $\Omega_\alpha(F_\bullet), \Omega_\beta(G_\bullet), \Omega_p(H_\bullet)$. Note that $V_1 \in \mathbb{P}(W/S) \cap \Omega_\alpha(F_\bullet)$, $V_2 \in \mathbb{P}(W/S) \cap \Omega_\beta(G_\bullet)$. Let us denote by $V_3 \subset W$ any subspace of dimension l which contains S and $W \cap H_{n-l-p+1}$. By the definition $S \subset V_3 \subset W$ so $V_3 \in \mathbb{P}(W/S)$. Note also that $V_3 \in \Omega_p(H_\bullet)$ since $V_3 \cap H_{n-l+1-p} \supset W \cap H_{n-l+1-p}$ and the latter has dimension 1. We conclude that $V_3 \in \mathbb{P}(W/S) \cap \Omega_p(H_\bullet)$ and the claim follows.

□

Theorem 4.2. *Pick $\lambda \in P(l \times k)$ and $0 \leqslant p \leqslant n - l$. Then we have*

$$\sigma_p \cdot \sigma_\lambda = \sum_{\substack{\mu, |\mu| = |\lambda| + p \\ n-l \geqslant \mu_i \geqslant \lambda_i \geqslant \mu_{i+1}}} \sigma_\mu + q \sum_{\substack{\nu, |\nu| = |\lambda| + p - n \\ \lambda_i - 1 \geqslant \nu_i \geqslant \lambda_{i+1} - 1 \geqslant 0}} \sigma_\nu. \tag{5}$$

Proof. Directly follows from the classical Pieri formula (see proposition 1.4) and lemma 4.1. □

Remark 4.3. Note that the first sum in (5) is taken over all μ that can be obtained from λ by adding p boxes with no two in the same column and the second sum is zero if $\ell(\lambda) < l$ and otherwise is taken over all ν such that ν that can be obtained from $(\lambda_1 - 1, \ldots, \lambda_l - 1)$ by adding $l+p-n$ boxes with no two in the same column.

4.2 Quantum Giambelli

We can now prove the quantum version of the Giambelli theorem.

Theorem 4.4 ([Be]). *If λ is a partition contained in $l \times k$ rectangle then the Schubert class $\tilde{\sigma}_\lambda$ in $QH^* \operatorname{Gr}(l, V)$ is given by $\tilde{\sigma}_\lambda = \det(\tilde{\sigma}_{\lambda_i + j - i})$, where $\tilde{\sigma}_i = 0$ for $i < 0$ or $i > k$.*

Proof. Let us prove that if $0 \leqslant i_j \leqslant k$ for $1 \leqslant j \leqslant l$ then $\tilde{\sigma}_{i_1} \cdot \tilde{\sigma}_{i_2} \cdot \cdots \cdot \tilde{\sigma}_{i_l} = (\sigma_{i_1} \cdot \sigma_{i_2} \cdot \ldots \cdot \sigma_{i_l}) \otimes 1$, i.e. no q-terms show up when the first product is expanded in the quantum ring. Using theorem 4.2 we easily prove by induction on j that the expansion of $\tilde{\sigma}_{i_1} \cdot \tilde{\sigma}_{i_2} \cdot \ldots \cdot \tilde{\sigma}_{i_j}$ involves no q-terms and no partitions of length greater than j.

Another proof of this uses lemma 3.7 and the fact that expansion of $\sigma_\lambda \cdot \sigma_\mu$ contains no terms of length greater than $l(\lambda) + l(\mu)$. This fact follows from Littlewood-Richardson rule. Here we need this fact only for $\mu = (p) = (p, 0, 0, \ldots, 0)$, so it follows from the usual Pieri rule.

Since $\det(\tilde{\sigma}_{\lambda_i + j - i}) = \sum_{\pi \in S_n} (-1)^{\operatorname{sgn}(\pi)} \prod_{i=1}^n \tilde{\sigma}_{\lambda_i + \pi(i) - i}$ we deduce that

$$\det(\tilde{\sigma}_{\lambda_i + j - i}) = \det(\sigma_{\lambda_i + j - i}) \otimes 1 = \sigma_\lambda \otimes 1 = \tilde{\sigma}_\lambda$$

. □

5 Presentation via generators and relations

Let $A := QH^*(X, \mathbb{Z})$. We define $c_i \in A$ as $c_i = \tilde{\sigma}_{1^i}$. For $p \geqslant 1$ we define $\tilde{\sigma}_p = \det(c_{1+j-i})_{1 \leq i, j \leq p}$. For $p < n$ using Lemma 3.7 or Theorem 4.2 we have $\tilde{\sigma}_p = \sigma_p \otimes 1$. Hence this definition agrees with previous definition of $\tilde{\sigma}_p$. Using this definition of $\tilde{\sigma}_p$ and first row decomposition of determinant we get

$$\sum_{i=1}^{l} (-1)^i \tilde{\sigma}_{m-i} c_i = 0 \tag{6}$$

Lemma 5.1. *We have $\tilde{\sigma}_n = (-1)^{l-1} q$.*

Proof. Using quantum Pieri and (6) we get

$$\tilde{\sigma}_n = (-1)^{l-1}\tilde{\sigma}_k\tilde{\sigma}_{1^l} = (-1)^{l-1}q.$$

□

Proposition 5.2. *We have an isomorphism of $\mathbb{Z}$-algebras*

$$H^*(\mathrm{Gr}(l,V),\mathbb{Z}) \cong \mathbb{Z}[x_1,\ldots,x_l,q]/(y_{k+1},\ldots,y_{n-1},y_n+(-1)^l q)$$

given by $c_i \mapsto x_i$, here $y_p := \det(x_{1+j-i})_{1\le i,j\le p}$.

Proof. Let

$$B := \mathbb{Z}[x_1,\ldots,x_l,q]/(y_{k+1},\ldots,y_{n-1},y_n+(-1)^l q)$$

where $\sigma_p = \det(c_{1+j-i})_{1\le i,j\le p}$. Using lemma 5.1 we get well-defined map $\phi\colon B \to A$, $\phi(x_i) = c_i$. Ring A is a free $\mathbb{Z}[q]$-module. A standard algebraic lemma says that a map $\psi\colon M \to N$ of $\mathbb{Z}[q]$-modules with N free is an isomorphism if and only if induced map $\psi'\colon M/qM \to N/qN$ is an isomorphism. Hence it is enough to prove that $\phi'\colon B/qB \to A/qA$ is an isomorphism. We have $B/qB = \mathbb{Z}[x_1,\ldots,x_l]/(y_{k+1},\ldots,y_n)$, $A/qA = H^*(X,\mathbb{Z})$, $\phi'(x_i) = c_i$. Using presentation of $H^*(X)$ via generators and relations (theorem 1.8) we deduce that ϕ' is an isomorphism. □

References

[Be] A. Bertram, *Quantum Schubert calculus*, Adv. Math. **128** (1997), 289-305.

[Bu] A. S. Buch, *Quantum cohomology of Grassmannians.*

[Gr] A. Grothendieck, *Thechniques de construction et téoremes d'existence en geometrie algebrique IV: Les schemas de Hilbert, Séminaire Bourbaki* **221** (1960/1961).

[N] N. Nitsure, *construction of Hilbert and quot schemes.*